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Redundant Reaction Wheel Torque Distribution Yielding Instantaneous L_2 Power-Optimal Attitude Control

Hanspeter Schaub*
University of Colorado at Boulder, Boulder, Colorado 80309
and
Vaios J. Lappas†
Surrey Space Centre, Guildford, Surrey, GU2 7XH, United Kingdom

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The attitude-control problem of a rigid spacecraft containing a redundant set of reaction wheels is investigated. Particularly with small spacecraft the available power is very limited due to the small surface area to radiate excess heat. A power-optimal reaction wheel motor torque distribution strategy is developed that minimizes the instantaneous electrical power requirements. Power regeneration from slowing down the wheels is not considered in this work. The new torque distribution is developed as a modification to the traditional minimum-torque solution. Degenerate conditions in which at least one rotor has zero speed are investigated, as well as particular symmetric wheel speed configurations. The new control is able to reduce the amount of mechanical power and energy required by about 10–20%, while only marginally increasing the average required torque.

I. Introduction

ACTUATION methods to control the orientation of a spacecraft typically fall into the categories of fuel consuming thrusters [1–3], internal momentum exchange devices requiring electrical power [4,5], or external environmental influences such as the gravity gradient, atmospheric, or magnetic torques [6–8]. The attitude control of spacecraft continues to be a rich area of research with many new issues being investigated. While some papers focus on developing robust adaptive attitude-control strategies using thrusters [9], this paper focuses on the spacecraft attitude control using momentum exchange devices. In particular, this paper does not develop a new attitude-control algorithm. Rather, it investigates how to effectively map a required control torque from a given attitude-control law to the set of reaction wheel motor torques. The spacecraft is assumed to contain a redundant cluster of reaction wheels (RWs). These RW clusters are also referred to as a reaction wheel assembly (RWA), or systems of momentum wheels. The RWs exert a torque onto the spacecraft by spinning up or down the flywheel [10]. These mechanically simple devices are limited in the amount of torque they can produce, and have rotational speed limits to which the flywheel can be spun up to. Having four or more RWs allows for full three-axis attitude control even if a particular RW has a mechanical failure. It is possible to control the attitude motion using only three orthogonal RWs and use the fourth only in case of a failure, or to use all RWs at all times. The later solution will result in more wear on the fourth wheel, but can yield reduced torque requirements.

Given a particular attitude-control strategy, this redundant RW setup yields an infinity of possible RW motor torque solutions that all stabilize the attitude motion. To keep the RW motors as small as possible, a simple solution is to determine the traditional minimum norm RW motor torque solution [10]. Using all RWs at once, this strategy results in smaller individual wheel torques than the minimum three wheel solution. The focus of this paper is to investigate alternate RW motor torque distributions that do not minimize the motor torques, but rather the power required to produce these torques. As with the minimum-torque solution, the optimization is not performed across the entire attitude maneuver. Instead, the minimization occurs over an instant of time where a torque solution is determined that requires the smallest instantaneous power level. The attitude-control method is not changed in this process. In fact, the presented torque-distribution strategies can be applied to any redundant RW attitude-control strategy and the results are not tied to a particular attitude-control law. Of interest is how such a power-minimizing torque-distribution strategy will result in reduced maneuver energy requirements.

A particular motivation of this work is the attitude control of small satellites which contains its own set of challenges. The small spacecraft are very limited in the amount of onboard propellant, and thus cannot afford to use this fuel to perform the attitude control [11,12]. Instead the use of momentum wheels such as RW or control-moment gyroscopes (CMGs) is considered as a more energy efficient attitude-control method [13,14]. However, note that the RW and CMG devices of a small satellite typically operate at much higher spin rates than those of a more typically sized spacecraft. This is important when considering instantaneous power usage because the power is proportional to the rotor speeds. Further, the amount of electrical power that a small satellite can produce is very limited. Because of the small size there is little surface area to reject the excess heat. The focus of this paper is the attitude control of a spacecraft which is limited in its available power and energy. While this work is motivated by the small satellite attitude problem, the results are general and can be applied to general spacecraft containing a redundant set of RWs.

The design and control of RW clusters has been discussed in previous publications, but none offer a locally power-optimal feedback control law. For example, [15] discusses the optimal RW alignment to produce optimal RW torque or power solutions. Vadali and Junkins [16] discuss optimal control solutions that minimize various performance aspects across a maneuver. Being an optimal control solution, such control torque calculations require knowledge of both the initial and final attitude states. In contrast, the novel RW motor torque distribution strategies developed in this paper are applicable to feedback control methods that require only the instantaneous attitude states, and not the knowledge of the entire maneuver. The torque solutions do not lead to maneuver optimal solutions, but provide simpler to implement feedback control strategies. The

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*Associate Professor, H. Joseph Smead Fellow, Aerospace Engineering Sciences Department, Colorado Center for Astrodynamics Research. Associate Member AIAA.

†Lecturer in Attitude/Orbit Control Systems, School of Electronics and Physical Sciences. Senior Member AIAA.

power-optimal RW spacecraft attitude control with a single RW is discussed by Skaar and Kraige [17,18]. However, here too optimal attitude trajectory paths are determined assuming the initial state errors are given. Tsiotras et al. [19] studies the attitude control of a spacecraft with a redundant set of RWs where the flywheels are also used as an energy storage mechanism. The null-space of the RWA is exploited to determine motor torques that satisfy both attitude stability conditions, as well as power generation requirement. In contrast, this paper investigates how the RW null-space can be used to modify the traditional minimum-torque solution and yield a power-optimal torque solution at that instant.

The mechanical power required to implement a particular motor torque is proportional to the wheel speed [10]. Thus, if all wheels are at rest relative to the spacecraft, then the associated mechanical power required is zero. References such as [20,21] address power requirement concerns by having attitude-control strategies that keep the RW speeds small. The RWA null-space is used to drive the wheel speeds to a lower value over time. This results in smaller mechanical power requirements, but involves RW speed minimizations that occur over a maneuver. The new torque-distribution strategy presented in this paper is quite different from such RW spin minimization strategies. At a particular instant of time, the wheel speeds are given and cannot be chosen. Given general nonzero spin rates, this paper investigates how the RW motor torque null-space can be used to yield a locally power-optimal RW motor torque solution. Attitude-control strategies that implement lower RW speeds to reduce power requirement can be used in conjunction with the locally power-optimal results discussed in this paper. For example, the zero RW speed crossing can cause issues for RWAs due to stickage at low speeds. As a result, the RWs can be setup to operate at a nonzero spin rate. If this nominal spin rate can be reduced, then the nominal electrical power required for a particular attitude maneuver is also reduced. The presented torque-distribution strategy can be employed in addition to such nominal wheel speed considerations.

The paper is set up as follows. First, the equations of motion of a rigid spacecraft containing N RWs are developed, and the notation is explained. Next, the attitude feedback control solution is developed, which uses the minimal RW motor torque distribution. Finally, the analytical closed-form solution of the locally power-optimal redundant RW control is developed. Degenerate conditions where some of the RWs have zero spin rates are investigated. Numerical simulations illustrate the performance of the new locally power-optimal feedback control and compare it to the torque-optimal and three-rotor solutions. Of interest is how much the instantaneous power and total energy expenditure is reduced by this alternate RW motor torque strategy.

II. Problem Statement

The spacecraft is assumed to be composed of a rigid body $\mathcal{B}$ containing N variable-speed RWs. The spacecraft body-fixed coordinate frame is given by $\mathcal{B}$: $\{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$. The orientation of each RW is defined through the body-fixed wheel frames $\mathcal{G}_i$: $\{\hat{\boldsymbol{g}}_{s_i}, \hat{\boldsymbol{g}}_{t_i}, \hat{\boldsymbol{g}}_{g_i}\}$ illustrated in Fig. 1. Because of the wheel symmetry about $\hat{\boldsymbol{g}}_{s_i}$ the actual orientation of the wheel body is not required. The disk is spinning with a speed Ω_i about the spin axis $\hat{\boldsymbol{g}}_{s_i}$. The motor torque $\boldsymbol{u}_{s_i}$ acts about the $\hat{\boldsymbol{g}}_{s_i}$ axis to accelerate the wheel as required by the attitude-control law.

Schaub and Junkins [10] develop the attitude equations of motion for such a system. The differential equations of motion are given by

$$[I]\dot{\boldsymbol{\omega}} = -[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - [\tilde{\boldsymbol{\omega}}][G_s]\boldsymbol{h}_s - [G_s]\boldsymbol{u}_s + \boldsymbol{L} \tag{1}$$

where $\boldsymbol{L}$ is an external torque vector, and $[\tilde{\boldsymbol{\omega}}]$ is defined as a matrix equivalent of a vector cross product using

$$[\tilde{\boldsymbol{\omega}}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{2}$$

To express the body angular velocity vector in spacecraft body frame $\mathcal{B}$ or wheel frame $\mathcal{G}_i$ vector components, the following notation is used:

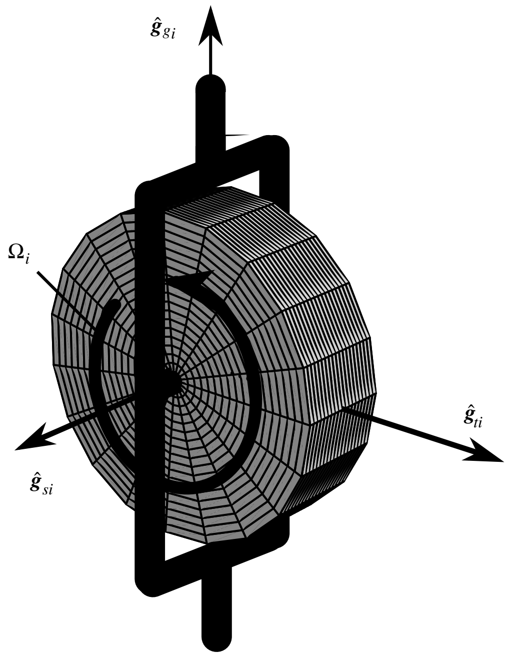


Fig. 1 Illustration of a RW coordinate frame.

$$\boldsymbol{\omega} = \omega_{s_i}\hat{\boldsymbol{g}}_{s_i} + \omega_{t_i}\hat{\boldsymbol{g}}_{t_i} + \omega_{g_i}\hat{\boldsymbol{g}}_{g_i} = \omega_1\hat{\boldsymbol{b}}_1 + \omega_2\hat{\boldsymbol{b}}_2 + \omega_3\hat{\boldsymbol{b}}_3 \tag{3}$$

The 3×3 matrix $[I]$ is the constant inertia system matrix defined as

$$[I] = [I_s] + \sum_{i=1}^{N}(J_{t_i}\hat{\boldsymbol{g}}_{t_i}\hat{\boldsymbol{g}}_{t_i}^T + J_{t_i}\hat{\boldsymbol{g}}_{g_i}\hat{\boldsymbol{g}}_{g_i}^T) \tag{4}$$

where $[I_s]$ is the inertia matrix of the rigid spacecraft itself. Because of symmetry the wheel principal inertias are given by $(J_{s_i}, J_{t_i}, J_{t_i})$. The wheel frames $\mathcal{G}_i$ are assumed to be principal coordinate frames for the RW disks such that the wheel inertias are defined through

$$[I_{W_i}] = J_{s_i}\hat{\boldsymbol{g}}_s\hat{\boldsymbol{g}}_s^T + J_{t_i}\hat{\boldsymbol{g}}_t\hat{\boldsymbol{g}}_t^T + J_{t_i}\hat{\boldsymbol{g}}_g\hat{\boldsymbol{g}}_g^T \tag{5}$$

Please note that this $[I]$ inertia matrix definition includes the inertia of the spacecraft, the $\hat{\boldsymbol{g}}_{t_i}$ and $\hat{\boldsymbol{g}}_{g_i}$ components of the wheel inertia, as well as the inertia terms since the RW center of masses is offset from the spacecraft body center of mass. The RW inertias J_{s_i} about the spin axis are subtracted out of this inertia matrix expression.

The N-dimensional torque vector $\boldsymbol{u}_s$ is the RW torque control vector and is defined as

$$\boldsymbol{u}_s = \begin{pmatrix} \vdots \\ u_{s_i} \\ \vdots \end{pmatrix} \tag{6}$$

where u_{s_i} are the ith RW motor torques defined through

$$u_{s_i} = J_{s_i}\left(\dot{\Omega}_i + \hat{\boldsymbol{g}}_{s_i}^T\dot{\boldsymbol{\omega}}\right) \tag{7}$$

The N-dimensional momentum vector $\boldsymbol{h}_s$ is defined as

$$\boldsymbol{h}_s = \begin{pmatrix} \vdots \\ J_{s_i}(\omega_{s_i} + \Omega_i) \\ \vdots \end{pmatrix} \tag{8}$$

Finally, the $3 \times N$ projection matrix $[G_s]$ is given by

$$[G_s] = \left[\hat{\boldsymbol{g}}_{s_1} \cdots \hat{\boldsymbol{g}}_{s_N}\right] \tag{9}$$

Numerically Eq. (1) requires that all vector components are taken with respect to the same coordinate frame before performing matrix algebra.

The rotational kinetic energy T of a rigid spacecraft with N RWs is given by [10]

$$T = \frac{1}{2}\boldsymbol{\omega}^T[I_s]\boldsymbol{\omega} + \frac{1}{2}\sum_{i=1}^{N} J_{s_i}(\Omega_i + \omega_{s_i})^2 + J_{t_i}\omega_{t_i}^2 + J_{t_i}\omega_{g_i}^2 \quad (10)$$

The kinetic energy rate, also known as the work rate or power equation, is found after differentiating Eq. (10), or simply by applying the work-energy-rate principle [10,22], to be

$$\dot{T} = \boldsymbol{\omega}^T \boldsymbol{L} + \sum_{i=1}^{N} \Omega_i u_{s_i} \quad (11)$$

The mechanical power equation to implement the ith motor torque is simply $\Omega_i u_{s_i}$ where Ω_i is the wheel angular velocity of the wheel relative to the spacecraft. The ω_{s_i} term drops out and does not contribute to the power expression. Therefore, in the absence of an external torque vector $\boldsymbol{L}$, the mechanical power P_i required by each RW motor is given by

$$P_i = \Omega_i u_{s_i} \quad (12)$$

Note that this is simply the mechanical RW power and does not take into consideration the power P_o required to operate the support electronics. This power bias to operate the wheels is nominally constant, and thus does not influence the power minimization algorithm.

III. Minimum-Torque Redundant Reaction Wheel Control Law

To control the spacecraft attitude, a feedback control law $\boldsymbol{u}_s$ is required to stabilize the spacecraft to a desired orientation. The following development does not depend on the specific type of attitude feedback control law that is chosen. Instead, all redundant RW feedback control strategies lead to an algebraically equivalent condition that maps the RW motor torques $\boldsymbol{u}_s$ to a required control torque $\boldsymbol{L}_r$. The novel content of this paper is how the $\boldsymbol{L}_r$ is mapped into $\boldsymbol{u}_s$ to provide the locally L_2 power-optimal solution.

To setup the redundant RW control problem, let $\boldsymbol{\sigma}$ be a set of modified Rodrigues parameters (MRPs) [10,23–26] which define the orientation of the body frame $\mathcal{B}$ with respect to a reference frame $\mathcal{R}$. The vector $\boldsymbol{\omega}$ is the body angular velocity of the spacecraft body, while $\boldsymbol{\omega}_r$ is the desired reference angular velocity vector. The angular velocity error vector $\delta\boldsymbol{\omega}$ can be defined as

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r \quad (13)$$

To develop a stabilizing feedback control law for this attitude trajectory tracking problem, the following positive definite Lyapunov function V can be used [1,10,25]:

$$V(\boldsymbol{\sigma}, \delta\boldsymbol{\omega}) = \frac{1}{2}\delta\boldsymbol{\omega}^T[I]\delta\boldsymbol{\omega} + 2K \ln(1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma}) \quad (14)$$

After setting the time derivative of V equal to the negative semidefinite function

$$\dot{V} = -\delta\boldsymbol{\omega}[P]\delta\boldsymbol{\omega} \quad (15)$$

and substituting the equations of motion in Eq. (1), the required RW motor torque vector is defined through the constraint:

$$[G_s]\boldsymbol{u}_s = K\boldsymbol{\sigma} + [P]\delta\boldsymbol{\omega} - [\tilde{\omega}]([I]\boldsymbol{\omega} + [G_s]\boldsymbol{h}_s - \boldsymbol{\omega}_r) - [I](\dot{\boldsymbol{\omega}}_r - \boldsymbol{\omega} \times \boldsymbol{\omega}_r) + \boldsymbol{L} = \boldsymbol{L}_r \quad (16)$$

The left hand side of Eq. (16) contains a projection matrix $[G_s]$ which maps the actual RW motor torques in the actual torque exerted onto the vehicle. The right hand side of Eq. (16) is the reference control torque $\boldsymbol{L}_r$ that is required by the chosen feedback control strategy,

$$[G_s]\boldsymbol{u}_s = \boldsymbol{L}_r \quad (17)$$

Note that while there are an infinity of $\boldsymbol{u}_s$ choices which produce the required torque, all control solutions will yield the same attitude closed loop dynamics with the same $\boldsymbol{\sigma}$ and $\delta\boldsymbol{\omega}$ time histories. However, the RW spin rates Ω_i will be different for different choices of the RW torques.

If the matrix $[G_s]$ is full rank then the RW cluster can produce the exact required control torque $\boldsymbol{L}_r$. If this projection matrix is not full rank, then $\boldsymbol{L}_r$ can only be partially produced. With RW clusters, the geometry of the spin axis is generally chosen such that the $\hat{\boldsymbol{g}}_{s_i}$ vectors span the three-dimensional space, and thus $[G_s]$ is full rank. Further, for the RW cluster control problem, $[G_s]$ is a constant matrix. If more than three RWs are employed, then the $[G_s]$ matrix contains a nonempty null-space, resulting in an infinity of u_{s_i} combinations that produce the required control torque $\boldsymbol{L}_r$.

Please note that *all* RW cluster control formulations can be written in the compact form shown in Eq. (17). If a different control strategy is chosen, then only the required torque definition of $\boldsymbol{L}_r$ changes. For redundant RW setups, the typical RW motor torque strategy employed seeks the minimum norm solution $\boldsymbol{u}_s^*$ which leads to the smallest absolute motor torques. This solution is given by

$$\boldsymbol{u}_s^* = [G_s]^T([G_s][G_s]^T)^{-1}\boldsymbol{L}_r \quad (18)$$

This solution is convenient when the RW motor torque limits are of concern. This is the case when the RW cluster is controlling the attitude of a large and massive spacecraft.

Of interest is exploring an alternate method of mapping the required control torque $\boldsymbol{L}_r$ into the RW motor torque vector $\boldsymbol{u}_s$. Instead of minimizing the instantaneous torque requirement, the RW motor power requirements are investigated.

IV. Power-Optimal Control Formulation

Small satellites are very limited in the amount of electrical power that they can produce or the amount of energy that they can store. The SNAP-I nanosatellite discussed in [27] is an example of a power-limited small spacecraft. Such spacecraft concepts are limited in how much electrical power they can provide while radiating out excess thermal energy through the small spacecraft surface area. Lappas et al. [13] discuss experimental results of a cluster of miniature CMG devices to control the small spacecraft orientation. A key concern in this study is the peak power requirement and the total energy consumed for a maneuver.

The RW cluster control law solution in Eq. (18), which minimizes the instantaneous motor torques, may not be the ideal solution for a small satellite with strong power and energy consumption limitations. This section investigates an alternate method of mapping the required control torque $\boldsymbol{L}_r$ to the RW control torques $\boldsymbol{u}_s$ in Eq. (17). Note that both control strategies use the same Lyapunov function in Eq. (14), and both have the same required torque $\boldsymbol{L}_r$ expression; they differ only in the resulting motor torque computation. Let R be the rank of the $3 \times N$ projection matrix $[G_s]$, while $M = N - R$ is the degree of redundancy in the RW cluster. The minimum RW motor torque solution $\boldsymbol{u}^*$ is only one of an infinity of solutions. Let the general motor torque vector be expressed as

$$\boldsymbol{u}_s = \boldsymbol{u}_s^* + [N]\boldsymbol{t} \quad (19)$$

where $[N]$ is the $N \times M$ the null-space matrix of $[G_s]$ satisfying

$$[G_s][N] = [0_{3\times M}] \quad (20)$$

The vector $\boldsymbol{t}$ contains the M null-space scaling parameters through

$$\boldsymbol{t} = (t_1 \quad \cdots \quad t_M)^T \quad (21)$$

For a given RW cluster the goal is to find the null-space scaling parameters t_i such that the instantaneous power consumption is minimized. The total instantaneous mechanical power P required is given by

$$P = \sum_{i=1}^{N} \Omega_i u_{s_i} = \sum_{i=1}^{N} P_i \quad (22)$$

However, note that the P_i components can be positive or negative. A positive power P_i means that the ith RW device requires a power input to achieve the maneuver. A negative power implies that the RW could return mechanical energy to the cluster. For example, consider the case where the spin wheel must be decelerated. Instead of applying brakes that would convert the mechanical spin energy into heat, it is possible to use a dynamo device that decelerates the wheel and converts its mechanical energy into stored electrical energy. This retrieved energy could be used to accelerate other wheels. In this case it makes sense to try to minimize the total instantaneous mechanical power usage in Eq. (22). Following such a research path the energy retrieval efficiency must be taken into account. Such optimal solutions can be determined numerically, but are very challenging to develop analytically. Instead, this paper focuses on the simpler situation where no energy retrieval mechanism is present. In this case a different cost function must be used to account for both acceleration and deceleration contributing to the total electrical power requirement. The benefit of this approach is that analytical torque solutions can be obtained.

Let $\boldsymbol{P} = (P_1 \quad \cdots \quad P_N)^T$ be a vector containing the RW powers P_i. Using Eq. (12), the list of RW powers $\boldsymbol{P}$ is expressed as

$$\boldsymbol{P} = [\Omega]\boldsymbol{u}_s \quad (23)$$

where the diagonal matrix $[\Omega]$ is defined as

$$[\Omega] = \mathrm{diag}(\Omega_i) \quad (24)$$

Let the positive cost function J be defined in terms of the L_2 norm of $\boldsymbol{P}$:

$$J = \frac{1}{2}(|\boldsymbol{P}|_2)^2 = \frac{1}{2}\sum_{i=1}^{N} P_i^2 = \frac{1}{2}\boldsymbol{P}^T\boldsymbol{P} \quad (25)$$

This cost function takes into account that both acceleration and deceleration of RWs requires electrical power. Next the torque vector $\boldsymbol{u}_s$ must be found which minimizes this cost function. Using Eq. (19) and (23) the cost function J is rewritten as

$$J = \frac{1}{2}\Big([\Omega](\boldsymbol{u}_s^* + [N]\boldsymbol{t})\Big)^T\Big([\Omega](\boldsymbol{u}_s^* + [N]\boldsymbol{t})\Big) \quad (26)$$

A necessary condition for a minimum of J with respect to the null-space scaling parameter is

$$\frac{\partial J}{\partial \boldsymbol{t}} = \Big([\Omega](\boldsymbol{u}_s^* + [N]\boldsymbol{t})\Big)^T[\Omega][N] = 0 \quad (27)$$

Carrying out the matrix algebra leads to

$$\underbrace{[N]^T[\Omega]^2[N]}_{[A]}\boldsymbol{t} = -[N]^T[\Omega]^2\boldsymbol{u}_s^* \quad (28)$$

Before solving for $\boldsymbol{t}$ the invertibility of $[A]$ must be investigated. The null-space matrix $[N]$ is expressed using the M-dimensional vectors $\boldsymbol{n}_i$ as

$$[N] = \begin{bmatrix} \boldsymbol{n}_1^T \\ \vdots \\ \boldsymbol{n}_N^T \end{bmatrix} \quad (29)$$

Note that none of the $\boldsymbol{n}_i$ vectors are a zero vector. The $M \times M$ matrix $[A]$ is then written as

$$[A] = \sum_{i=1}^{N} \Omega_i^2 \boldsymbol{n}_i \boldsymbol{n}_i^T \quad (30)$$

Because $[N]$ has rank M through its definition as the null-space matrix of $[G_s]$, the rank of $[A]$ is also M if the RW spin rates are nonzero with $\Omega_i^2 > 0$. In fact, the matrix $[A]$ has rank M and is invertible if at least M RWs have a nonzero spin rate. For example, if there are four RWs on the spacecraft, then the null-space $[N]$ of $[G_s]$ is a 4×1 matrix with $M = 1$. Because $[N]$ cannot contain columns or rows of zeros, all components of $[N]$ are nonzero in this case. Here $[A]$ is invertible as long as at least one RW has a nonzero speed. If the spacecraft has five RWs, then at least two RWs will have to have nonzero spin rates. If $[A]$ is invertible, then the optimal null-space scaling parameter vector $\hat{\boldsymbol{t}}$ is given by

$$\hat{\boldsymbol{t}} = -\Big([N]^T[\Omega]^2[N]\Big)^{-1}[N]^T[\Omega]^2\boldsymbol{u}_s^* \quad (31)$$

Setting $\partial J/\partial \boldsymbol{t} = \boldsymbol{0}$ is only a necessary condition for the power-optimal solution. To guarantee a minimum-power solution $\partial J^2/\partial \boldsymbol{t}^2 > 0$ must be a positive definite matrix. Differentiating Eq. (27) with respect to $\boldsymbol{t}$ yields

$$\frac{\partial J^2}{\partial \boldsymbol{t}^2} = [N]^T[\Omega]^2[N] \quad (32)$$

Using the $[N]$ definition in Eq. (29) this is rewritten as

$$\frac{\partial J^2}{\partial \boldsymbol{t}^2} = \sum_{i=1}^{N} \Omega_i^2 \boldsymbol{n}_i \boldsymbol{n}_i^T \quad (33)$$

which yields a positive definite matrix by inspection for the general case with $\Omega_i \neq 0$. Thus the solution in Eq. (31) provides the null-space scaling parameters yielding a minimum instantaneous power control.

What occurs if the $[A]$ matrix is not invertible? First, consider the simple case where all the RWs are at rest with $\Omega_i = 0$. Studying Eq. (26) it is apparent that the power cost function is zero regardless of which torque solution is used. Any torque solution in Eq. (19) would provide a power-optimal solution. In this case it would make sense to simply use the minimum-torque solution $\boldsymbol{u}_s^*$ and set $\hat{\boldsymbol{t}} = 0$.

Next the scenario is investigated where some Ω_i are nonzero, yet the $[A]$ matrix is not full rank. Let R be the number of nonzero RW spin rates Ω_i, where $R < M$ to guarantee that $[A]$ is not invertible. Without loss of generality let us assume that only the first R rotors have nonzero Ω_i. Equation (28) is satisfied if a vector $\boldsymbol{t}$ is chosen such that

$$\boldsymbol{n}_i^T\boldsymbol{t} = -u_{s_i}^* \quad \text{for } i = 1, \cdots, R \quad (34)$$

Using Eq. (30) the power-optimal scaling parameter condition in Eq. (28) is rewritten as

$$\begin{aligned}[A]\boldsymbol{t} &= (\Omega_1^2\boldsymbol{n}_1\boldsymbol{n}_1^T + \cdots + \Omega_R^2\boldsymbol{n}_R\boldsymbol{n}_R^T)\boldsymbol{t} \\ &= -\Omega_1^2\boldsymbol{n}_1 u_{s_1}^* - \cdots - \Omega_R^2\boldsymbol{n}_R u_{s_R}^* \end{aligned} \quad (35)$$

where $u_{s_i}^*$ is the ith components of $\boldsymbol{u}_s^*$. Because the square matrix $[A]$ is not full rank in this scenario, it is not possible to solve this equation for a unique $\boldsymbol{t}$. Instead, there are an infinity of scaling parameters that yield the desired power-optimal solution. A simple solution to Eq. (35) is to determine the minimum norm solution to $\boldsymbol{t}$. Let the $R \times M$ matrix $[\mathcal{N}]$ be defined as

$$[\mathcal{N}] = \begin{bmatrix} \boldsymbol{n}_1^T \\ \vdots \\ \boldsymbol{n}_R^T \end{bmatrix} \quad (36)$$

and $\boldsymbol{\mathcal{U}}_s = (u_{s_1}^* \quad \cdots \quad u_{s_R}^*)^T$, then the desired null-space scaling parameter vector $\boldsymbol{t}$ is determined using

$$\hat{\boldsymbol{t}} = -\mathcal{N}^T([\mathcal{N}][\mathcal{N}]^T)^{-1}\boldsymbol{\mathcal{U}}_s \quad (37)$$

for this degenerate scenario with an infinity of solutions.

While Eq. (37) provides an analytical solution for $\hat{\boldsymbol{t}}$, in practice the wheel speeds are rarely perfectly zero. Equation (37) could be implemented by using a finite zero-speed deadband to determine which wheels are effectively at rest. Because the power required for a zero-speed wheel is zero, having a small nonzero speed with the zero deadband calculation results in a very small power-optimality error. Other numerical options to invert a nonfull rank $[A]$ matrix include a singularity robust inverse [28], or invert only the nonzero singular values as employed by Hall and Ford in their control-moment gyroscope rate computation [29]. Minimal power-optimality errors would result from any of these approximate numerical methods because the wheel power approaches zero as Ω_i approaches zero.

V. Special Configurations

This section investigates particular special RW speed conditions and spin axis alignments. Of interest is how different the instanteneous power-optimal solution is from the torque-optimal solution $\boldsymbol{u}^*$. This is achieved by investigating null-space scaling parameter set $\hat{\boldsymbol{t}}$.

A. Equal RW Wheel Speeds

Consider the scenario where the N RWs have identical wheels speeds $\Omega_i = \Omega$. In this scenario the $N \times N$ matrix $[\Omega]$ is expressed as

$$[\Omega] = \Omega[I_{N\times N}] \tag{38}$$

Substituting the minimum-torque solution in Eq. (18) and (38) into the $\hat{\boldsymbol{t}}$ solution in Eq. (31) yields

$$\hat{\boldsymbol{t}} = -([N]^T[N])^{-1}[N]^T[G_s]^T([G_s][G_s]^T)^{-1}\boldsymbol{L}_r = 0 \tag{39}$$

Because of the null-space property where $[G_s][N] = 0$, note that for this equal RW speed configuration the null-space parameter set $\hat{\boldsymbol{t}}$ is always zero. This result is true regardless of the number of RWs, or their choice in body-fixed spin axes. Thus, in this configuration the torque-optimal and power-optimal attitude-control solution to the control constraint in Eq. (17) are identical.

B. Traditional Four RW Setup

A popular redundant four RW configuration used in several analytical attitude and control studies [10,19] has three axis aligned with the spacecraft body principal body frame axis $\hat{\boldsymbol{b}}_i$, with the fourth wheel diagonally aligned with the first three as illustrated in Fig. 2. In this scenario the four RW spin axis $\hat{\boldsymbol{g}}_{s_i}$ are set up as follows in spacecraft body frame $\mathcal{B}$ coordinates:

$$\hat{\boldsymbol{g}}_{s_1} = {}^{\mathcal{B}}\begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \hat{\boldsymbol{g}}_{s_2} = {}^{\mathcal{B}}\begin{bmatrix}0\\1\\0\end{bmatrix}$$
$$\hat{\boldsymbol{g}}_{s_3} = {}^{\mathcal{B}}\begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \hat{\boldsymbol{g}}_{s_4} = \frac{1}{\sqrt{3}}{}^{\mathcal{B}}\begin{bmatrix}1\\1\\1\end{bmatrix} \tag{40}$$

The 1×4 null-space matrix $[N]$ of $[G_s]$ is expressed as

$$[N] = [-\tfrac{1}{\sqrt{3}} \quad -\tfrac{1}{\sqrt{3}} \quad -\tfrac{1}{\sqrt{3}} \quad 1]^T \tag{41}$$

First assume that the RW spin speeds are given by Ω_1, Ω_2, Ω_3 and Ω_4, while the required torque is expressed as $\boldsymbol{L}_r = (L_1, L_2, L_3)$. Substituting the particular projection matrix in Eq. (40) and the associated null-space in Eq. (41) into the $\hat{\boldsymbol{t}}$ expression in Eq. (31) yields

$$\hat{\boldsymbol{t}} = \frac{L_1(-5\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + 3\Omega_4^2) + L_2(\Omega_1^2 - 5\Omega_2^2 + \Omega_3^2 + 3\Omega_4^2) + L_3(\Omega_1^2 + \Omega_2^2 - 5\Omega_3^2 + 3\Omega_4^2)}{2\sqrt{3}(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + 3\Omega_4^2)} \tag{42}$$

Note that if the ith wheel speed Ω_i is much larger than the remaining spin speeds, then the null-space correction factor will always approach

$$\hat{\boldsymbol{t}} \approx \frac{L_i}{2\sqrt{3}} \tag{43}$$

Note that surprisingly this result does not depend on the dominant Ω_i spin speed. Instead only the required torque vector $\boldsymbol{L}_r$ determines the torque null-space shift $\hat{\boldsymbol{t}}$. If the Ω_i are all set equal then $\hat{\boldsymbol{t}}$ becomes zero as expected from the equal RW speed discussion.

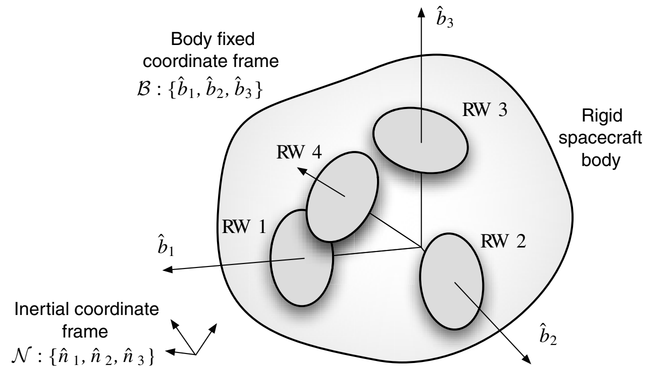


Fig. 2 Spacecraft illustration containing four RWs.

In the previous configuration the four wheel speeds were kept general. As a result, the RW cluster could contain a net angular momentum that would resist spacecraft motion and require larger control torques to overcome. Next, the configuration is investigated where the fourth wheel speed is set such that the total momentum of the RW cluster is zero initially. Assume that the first three RW spin rates are equal with $\Omega = \Omega_1 = \Omega_2 = \Omega_3$. The zero RW cluster momentum condition requires that

$$\Omega_4 = -\sqrt{3}\Omega \tag{44}$$

Substituting this Ω_4 condition into Eq. (42) yields the simple $\hat{\boldsymbol{t}}$ expression:

$$\hat{\boldsymbol{t}} = \frac{L_1 + L_2 + L_3}{4\sqrt{3}} \tag{45}$$

Note that in this configuration the null-space corrections are proportional to the $\boldsymbol{L}_r$ vector components L_i regardless of the Ω_i magnitudes.

VI. Numerical Simulations

Numerical simulations of a spacecraft containing four RWs are performed to compare the minimum-torque attitude-control solution in Eq. (18) to the minimum-power control proposed in Eq. (31). As illustrated in Fig. 2, the four RW spin axes are given in Eq. (40). This redundant RW configuration has the first three RW spin axes aligned with the principal spacecraft body axes, while a fourth wheel is aligned diagonally to the others. In this setup the loss of any RW can be compensated for by the remaining three RWs. To simulate the motion of a microsatellite, the spacecraft and RW inertias of the Tsinghua-1 [13,30] spacecraft built by Surrey Space Technologies are used:

$$[I] = \text{diag}(2.5, 2.5, 2.5)\ \text{kg m}^2$$

while the RW spin axis inertia is $J_s = 0.02\ \text{kg m}^2$. The maximum torque that these RWs can produce is 0.01 Nm.

The reference attitude is set to be that of the inertial frame $\mathcal{N}$, demonstrating the response of a regulator problem. Two different initial spacecraft state vectors are simulated. Scenario 1 has a large initial attitude error and is representative of doing an aggressive maneuver. Scenario 2 has a small orientation error allowing performance comparisons for small maneuvers. Both simulation scenarios use the same initial angular velocity vector.

$$\sigma(t_0) = (0.414, 0.300, 0.200) \quad \text{Scenario 1}$$

$$\sigma(t_0) = (0.000, 0.000, 0.000) \quad \text{Scenario 2}$$

$$\omega(t_0) = (0.03, 0.05, -0.01)\ \text{rad/s}$$

The power and torque performance of three control law cases are studied. Case 1 is the classical RW control strategy with three mutually-orthogonal RWs. This case uses only RWs 1–3 defined in Eq. (40) (RW spin axes aligned with body axes). Case 2 uses all four RWs in Eq. (40) and employs the traditional minimum-torque control solution in Eq. (18). Finally, case 3 also uses the four RWs of case 2 but uses the novel minimum-power motor torque distribution solution presented in this paper.

To provide fair comparisons between these three cases, the wheel speeds Ω_i of RWs 1–3 are initialized to a nonzero value (500 rpm), while the fourth wheel (if used) is initialized to zero. This results in the RW cluster having a nonzero angular momentum in all three cases. While a four-wheel configuration can be setup to have nonzero wheel speeds and a zero RWA momentum, the classical approach of using three orthogonal RW cannot operate with zero cluster momentum and nonzero wheel speeds. While an initially zero-momentum RWA configuration would make the spacecraft more agile and reduce the control torque requirements, it would not provide a meaningful comparison to the three RW setup in case 1. As such, the RWA initialization with $\Omega_1 = \Omega_2 = \Omega_3 \neq 0$ and $\Omega_4 = 0$ allows for reasonable performance comparisons between the three cases to illustrate the power savings possible while generating similar motor torque levels.

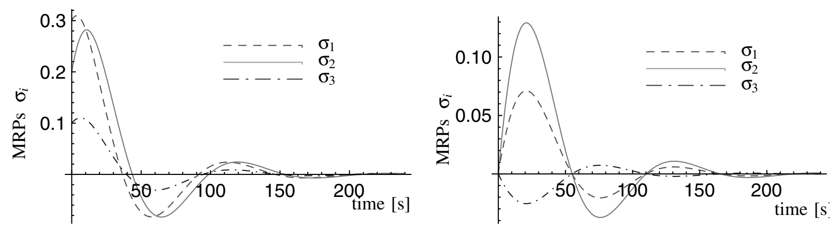
The control feedback gains are set to

$$K = 0.020\ \text{Nm} \qquad P = 0.045\ \text{Nms}$$

These gains are chosen such that the control torques of all three cases remain within the 0.01 Nm saturation bound. If more aggressive gains were chosen which saturate the RWs, then the power usage performance would be drastically influenced by how much saturation occurs and the resulting stability of the saturated response. The control law employed is guaranteed stable only for unsaturated control. By not having the control saturate, the power and torque usage performance comparisons are more informative.

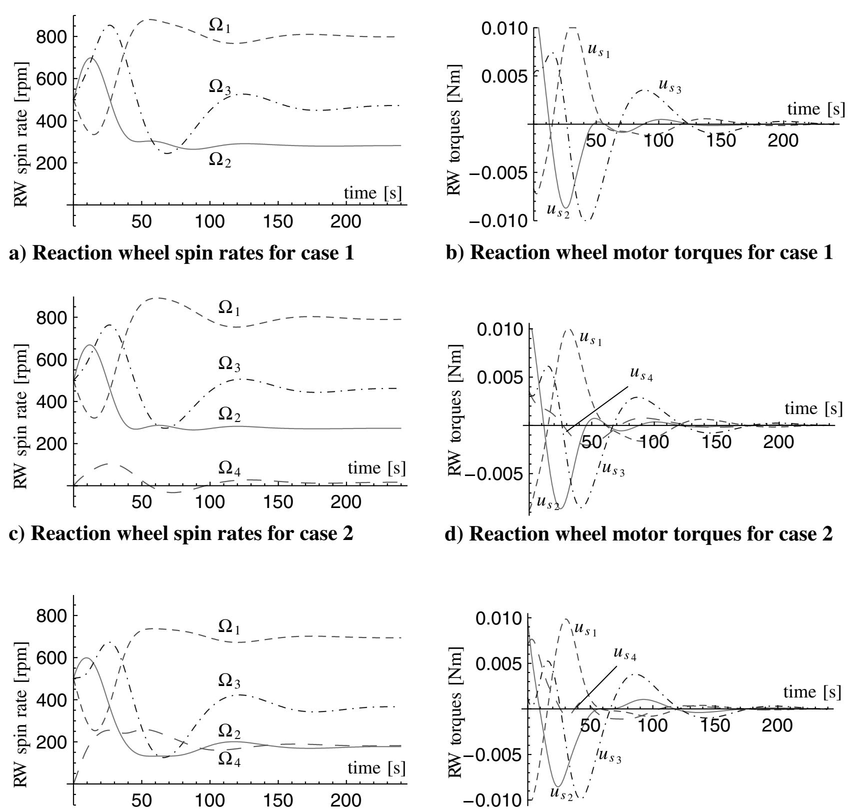
The numerical simulations are run for 240 s each. The attitude response for both large and initial small attitude errors are illustrated in Fig. 3. The spacecraft orientation errors always converge in the same manner because each case has the same closed loop equations of motion.

The RW spin rates and motor torque time histories are illustrated in Fig. 4. Note that the Ω_i of cases 1 and 2 do not differ very much. The



a) Scenario 1 b) Scenario 2

Fig. 3 Spacecraft attitude tracking errors.

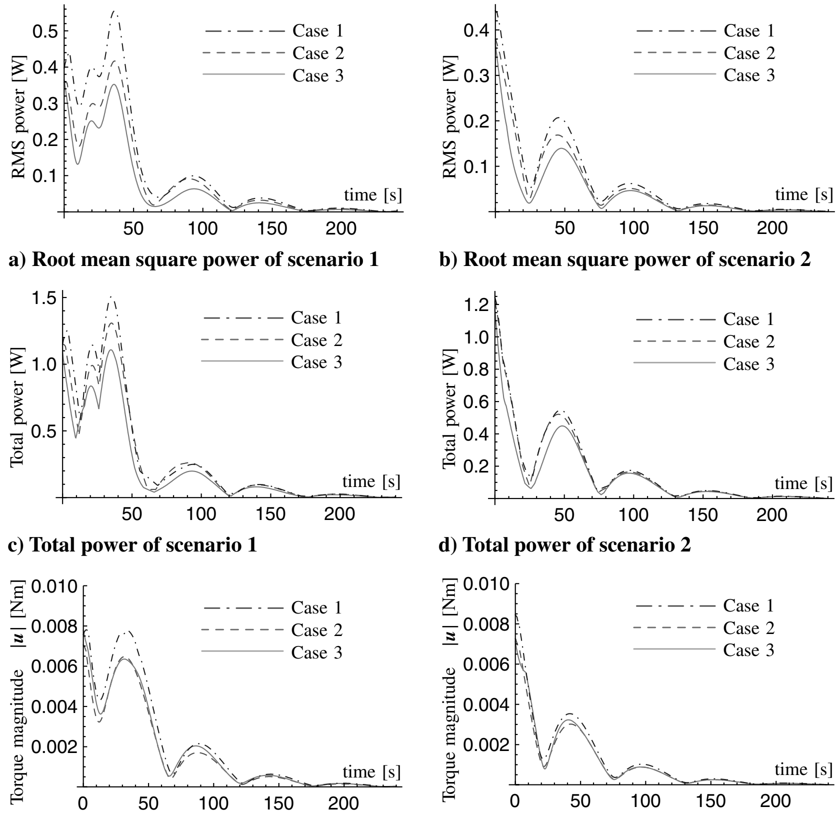


a) Reaction wheel spin rates for case 1 b) Reaction wheel motor torques for case 1

c) Reaction wheel spin rates for case 2 d) Reaction wheel motor torques for case 2

e) Reaction wheel spin rates for case 3 f) Reaction wheel motor torques for case 3

Fig. 4 Comparison of RW spin rates and motor torques for cases 1–3 of scenario 1.



a) Root mean square power of scenario 1
b) Root mean square power of scenario 2
c) Total power of scenario 1
d) Total power of scenario 2
e) Control norm of scenario 1
f) Control norm power of scenario 2

Fig. 5 Comparison of RW power and torque requirements for scenarios 1 and 2.

fourth wheel spin rate remains close to zero. The minimum-power control solution used in case 3 shows a larger usage of the fourth wheel. For all cases the RW motor torque remains within the 0.01 saturation bounds.

The electrical power and motor torque levels required for these three cases are compared in Fig. 5. Both the results of the large and small initial state errors of scenario 1 and 2 are illustrated next to each other. Figure 5a shows the root mean square power levels required across the cluster. The unique control solution for the three RW case requires the largest average power levels. Using the fourth wheel in case 2 reduced the overall power required noticeably with a redundant minimum-torque control solution. However, the proposed minimum-power solution reduces the average RW cluster power requirements even further. These results are mirrored for the small initial state error case and are shown in Fig. 5b. Note that the new minimum L_2 power solution does not guarantee that the resulting maneuver will use less energy than a maneuver employing the minimum-torque solution. The power minimization is only performed locally at the current time step. To be fair, the same can be said about the classical minimum-torque attitude-control solution. This control does not do maneuver-wide torque minimizations, but rather finds the smallest instantaneous torque solution to provide the required torque.

Figures 5c and 5d illustrate the total power used. This is computed using

$$P_{\text{total}} = \sum_{i=1}^{N} |P_i| \tag{46}$$

While the earlier root-mean-square (rms) power levels illustrate how well the $|\boldsymbol{P}|_2$ is minimized on average during the maneuver, it does not reflect the total power required at any particular time. Further, case 1 operates with one less RW than cases 2 and 3, and thus might still require less power for these maneuvers than the four-RW cases. As shown in Fig. 5c, the case 1 total power requirement is closer to the case 2 power requirement. The reduction in individual power of case 4 is partially offset by the requirement of an additional wheel. The new control in case 3 still requires the smallest instantaneous total power for these two maneuver cases. Similar results were found by varying the initial conditions.

While case 3 minimizes the instantaneous power requirement, it is expected that the required motor torques are increased. Figures 5e and 5f present the RW motor torque vector norm $|\boldsymbol{u}_s|$ for all three cases. Only operating with three RWs (case 1) routinely requires higher torque levels of the RW motors. However, while the required torques of case 3 are higher than those of case 2, the differences are very small. For a small increase in the motor torque levels a significant energy savings is achieved with the proposed minimum-power RW motor torque distribution solution.

The normalized root mean square energy requirements for case 1–3 maneuvers are listed in Table 1. The mechanical energy of accelerating the RW is accumulated across the maneuver and scaled by the RW inertial J_s [30]. Simply employing the previously at rest fourth RW during the maneuver (case 2) decreases the total power used by about 17–23%. The minimum-power RW control solution with the same initial conditions yields an average root mean square energy savings of 36–40% over case 1, and 22–23% compared with case 2. Note that these savings levels do not take into consideration the standby power levels needed simply to operate a RW. These are implementation specific and are often below 0.1 W for Tsinghua-1 [13,30]. The improvements of case 3 over case 2 are valid regardless of the standby power consumption because both cases employ all four RWs.

Table 1 Normalized rms energy ($\mathbf{J/kg \cdot m^2}$) usage comparisons

Scenario	Case 1	Case 2	Case 3
1	6779.30	5208.08 (−23.1%)	4035.55 (−40.5%, −22.5%)
2	3924.38	3226.40 (−17.8%)	2478.81 (−36.8%, 23.17%)

VII. Conclusion

The classical minimum wheel motor torque solution for a redundant cluster of RWs is revisited to examine instantaneous power-optimum RW motor torque distributions. The RW redundancy creates a null-space in the flywheel motor torque solution. An analytical solution is provided that determines the solution in the null-space that provides the smallest electrical power requirement using the L_2 norm at the current time step. If some RW speeds are effectively zero, then there are an infinity of power-optimal solutions. While analytical answers are provided for this situation, approximate numerical methods could also be employed with minimal impact on the instantaneous power requirement. As with the minimum-torque control solution to a redundant RW cluster, this control does not provide for global maneuver-wide optimal solutions. However, the new control strategy can typically provide around 10–20% energy savings for a minimal increase in the average torque used. The power savings of the four RW setup using the new minimum-power torque distribution over a three RW setup reached 30–40%. Future research could investigate L_1 power-optimal motor torque solutions where regenerated power of decelerated RWs can compensate for power required to accelerate other wheels.

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